Vectors

Norm and Angle

- Norm of a vector with 2 components: $\|[u_1 \quad u_2]\| = \sqrt{u_1^2 + u_2^2}$
- Norm of a vector with 3 components: $\|[u_1 \quad u_2 \quad u_3]\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$
- Angle θ of the vector $\boldsymbol{u} = [u_1 \quad u_2]$: $\|\boldsymbol{u}\| \cos\theta = u_1$, $\|\boldsymbol{u}\| \sin\theta = u_2$

Example Evaluate $\|[116 \quad -89]\|$ and the angle of $[116 \quad -89]$.

Solution:

$$\|[116 \quad -89]\| = \sqrt{116^2 + (-89)^2} = \sqrt{13456 + 7921} = \sqrt{\mathbf{21377}}$$

(First we find the reference angle ϕ)

$$\phi = \tan^{-1}\left(\left|\frac{-89}{116}\right|\right) = \tan^{-1}\left(\frac{89}{116}\right)$$

(Since 116>0 and –89<0, the angle of the vector, θ, is in the second quadrant and hence is given by $\theta = \pi - \phi$[remember to set your calculator in radian mode]):

$$\theta = \pi - \phi = \pi - \tan^{-1}\left(\frac{89}{116}\right) \approx \mathbf{2.4871\ rad}$$

Exercise Evaluate

- $\|[-5 \quad 10]\|$ [Answer: $5\sqrt{5}$]
- $\|[1 \quad -2 \quad 3]\|$ [Answer: $\sqrt{14}$]

Scalar multiplication

$$k[u_1 \quad u_2] = [ku_1 \quad ku_2]; \quad k[u_1 \quad u_2 \quad u_3] = [ku_1 \quad ku_2 \quad ku_3]$$

Property of Norm

$$\|k\boldsymbol{u}\| = |k|\,\|\boldsymbol{u}\|$$

Example For a vector $\boldsymbol{v}$ with norm 2, evaluate $\|3\boldsymbol{v}\| + \|-2\boldsymbol{v}\|$.

Solution:

(Use the following property of norm: $\|k\boldsymbol{v}\| = |k| \cdot \|\boldsymbol{v}\|$)

$$\|3\boldsymbol{v}\| + \|-2\boldsymbol{v}\| = |3| \cdot \|\boldsymbol{v}\| + |-2| \cdot \|\boldsymbol{v}\| = 3\|\boldsymbol{v}\| + 2\|\boldsymbol{v}\| = 5\|\boldsymbol{v}\| = 5 \times 2 = \mathbf{10}$$

Vectors

Addition of two-dimensional vectors

(Step 1) Break original vectors into x and y components: $\vec{\boldsymbol{A}} = A_x\vec{\boldsymbol{\imath}} + A_y\vec{\boldsymbol{\jmath}}$.

(Step 2) Add x pieces to x pieces to get R_x, add y pieces to y pieces to get R_y.

(Step 3) Apply Pythagorean formula to get the magnitude resultant R.

(Step 4) Use $\tan\theta = \frac{R_y}{R_x}$ together with the signs of R_x and R_y to get the angle θ.

Example For $\boldsymbol{u} = [7 \quad 2]$ and $\boldsymbol{v} = [-3 \quad 5]$, evaluate $3\boldsymbol{u} + 4\boldsymbol{v}$ and $\|5\boldsymbol{v} - 2\boldsymbol{u}\|$.

Solution:

$3\boldsymbol{u} + 4\boldsymbol{v} = 3[7 \quad 2] + 4[-3 \quad 5] = [3\times 7 \quad 3\times 2] + [4\times(-3) \quad 4\times 5]$

$=[21 \quad 6] + [-12 \quad 20] = [21 + (-12) \quad 6 + 20] = [\mathbf{9} \quad \mathbf{26}]$

$5\boldsymbol{v} - 2\boldsymbol{u} = 5[-3 \quad 5] - 2[7 \quad 2] = [5\times(-3) \quad 5\times 5] - [2\times 7 \quad 2\times 2]$

$=[-15 \quad 25] - [14 \quad 4] = [(-15) - 14 \quad 25 - 4] = [\mathbf{-29} \quad \mathbf{21}]$

$\|5\boldsymbol{v} - 2\boldsymbol{u}\| = \sqrt{(-29)^2 + 21^2} = \sqrt{841 + 441} = \sqrt{\mathbf{1282}} (\approx 35.8)$

Exercise Evaluate the following

- $[13 \quad -12 \quad -13] + 14[-1 \quad 1 \quad 1]$ [Answer: $[-1 \quad 2 \quad 1]$]
- $3[5 \quad -2] - [-1 \quad 8]$ [Answer: $[16 \quad -14]$]

Dot Product and Cross product

- Dot Product:

Vectors with 2 Components	Vectors with 3 Components
$[u_1 \quad u_2]\cdot[v_1 \quad v_2]$ $= u_1v_1 + u_2v_2$	$[u_1 \quad u_2 \quad u_3]\cdot[v_1 \quad v_2 \quad v_3]$ $= u_1v_1 + u_2v_2 + u_3v_3$
$\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\cos\theta \quad (0 \le \theta \le \pi)$	

- Cross Product: $[u_1 \quad u_2 \quad u_3] \times [v_1 \quad v_2 \quad v_3] = \begin{vmatrix} \vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

Example For the points $A = (1,0,1)$, $B = (0,2,2)$, and $C = (3,0,0)$, evaluate and simplify $\overrightarrow{\boldsymbol{AB}} \times \overrightarrow{\boldsymbol{AC}}$.

Solution:

$\overrightarrow{\boldsymbol{AB}} = [0-1 \quad 2-0 \quad 2-1] = [-1 \quad 2 \quad 1]$;

$\overrightarrow{\boldsymbol{AC}} = [3-1 \quad 0-0 \quad 0-1] = [2 \quad 0 \quad -1]$;

$$\overrightarrow{\boldsymbol{AB}} \times \overrightarrow{\boldsymbol{AC}} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ -1 & 2 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

$= (2)(-1)(\vec{\boldsymbol{i}}) + (1)(2)(\vec{\boldsymbol{j}}) + (-1)(0)(\vec{\boldsymbol{k}}) - (2)(2)(\vec{\boldsymbol{k}}) - (1)(0)(\vec{\boldsymbol{i}}) - (-1)(-1)(\vec{\boldsymbol{j}})$

$= -2\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} - 0\vec{\boldsymbol{k}} - 4\vec{\boldsymbol{k}} - 0\vec{\boldsymbol{i}} - 1\vec{\boldsymbol{j}} = -2\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}} - 4\vec{\boldsymbol{k}} = [\mathbf{-2} \quad \mathbf{1} \quad \mathbf{-4}]$

Example For $\boldsymbol{u} = [-2 \quad -3 \quad 0]$, $\boldsymbol{v} = [4 \quad 4 \quad -3]$ and $\boldsymbol{w} = [-3 \quad -3 \quad -1]$, evaluate $\|\boldsymbol{u}\|$, $\boldsymbol{u} \cdot \boldsymbol{v}$, $\boldsymbol{v} \times \boldsymbol{u}$ and $\boldsymbol{w} \cdot (\boldsymbol{v} \times \boldsymbol{u})$.

Solution:

$\|\boldsymbol{u}\| = \|[-2 \quad -3 \quad 0]\| = \sqrt{(-2)^2 + (-3)^2 + (0)^2} = \sqrt{4+9+0} = \sqrt{\mathbf{13}}$

$\boldsymbol{u} \cdot \boldsymbol{v} = (-2)(4) + (-3)(4) + (0)(-3) = -8 - 12 - 0 = \mathbf{-20}$

$$\boldsymbol{v} \times \boldsymbol{u} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ 4 & 4 & -3 \\ -2 & -3 & 0 \end{vmatrix}$$

$= (4)(0)(\vec{\boldsymbol{i}}) + (-3)(-2)(\vec{\boldsymbol{j}}) + (4)(-3)(\vec{\boldsymbol{k}}) - (4)(-2)(\vec{\boldsymbol{k}}) - (-3)(-3)(\vec{\boldsymbol{i}}) - (4)(0)(\vec{\boldsymbol{j}})$

$= 0\vec{\boldsymbol{i}} + 6\vec{\boldsymbol{j}} - 12\vec{\boldsymbol{k}} + 8\vec{\boldsymbol{k}} - 9\vec{\boldsymbol{i}} - 0\vec{\boldsymbol{j}} = -9\vec{\boldsymbol{i}} + 6\vec{\boldsymbol{j}} - 4\vec{\boldsymbol{k}} = [\mathbf{-9} \quad \mathbf{6} \quad \mathbf{-4}]$

$\boldsymbol{w} \cdot (\boldsymbol{v} \times \boldsymbol{u}) = [-3 \quad -3 \quad -1] \cdot [-9 \quad 6 \quad -4] = (-3)(-9) + (-3)(6) + (-1)(-4)$

$= 27 - 18 + 4 = \mathbf{13}$

Exercise

- Evaluate
  - $[-5 \quad -10] \cdot [3 \quad -2]$ [Answer: 5]
  - $[3 \quad 1 \quad 0] \cdot [-2 \quad 1 \quad -4]$ [Answer: -5]
  - $[2 \quad 2 \quad 0] \times [1 \quad 0 \quad 1]$ [Answer: $[2 \quad -2 \quad -2]$]
  - $[2 \quad -4 \quad 6] \times [4 \quad -2 \quad -1]$ [Answer: $[16 \quad 26 \quad 12]$]
  - $[1 \quad 2 \quad 1] \times [-1 \quad 1 \quad 1]$ [Answer: $[1 \quad -2 \quad 3]$]
  - $[12 \quad -8 \quad -2] \times [8 \quad -1 \quad 10]$ [Answer: $[-82 \quad -136 \quad 52]$]

  - ○ $[0 \quad 1 \quad 2] \times [-64 \quad 6 \quad 4]$ [Answer: $[-8 \quad -128 \quad 64]$]
- For $\boldsymbol{v} = [3 \quad 1 \quad 0]$ and $\boldsymbol{w} = [2 \quad -1 \quad 0]$, evaluate $\|\boldsymbol{v}\|$, $\|\boldsymbol{v}+\boldsymbol{w}\|$, $\boldsymbol{w}\cdot\boldsymbol{v}$, and $\|\boldsymbol{v}\times\boldsymbol{w}\|$.

  [Answer: $\|\boldsymbol{v}\| = \sqrt{10}$, $\|\boldsymbol{v}+\boldsymbol{w}\| = 5$, $\boldsymbol{w}\cdot\boldsymbol{v} = 5$, $\|\boldsymbol{v}\times\boldsymbol{w}\| = 5$]
- Evaluate $[1 \quad 1 \quad 1] \cdot ([1 \quad 1 \quad 2] \times [0 \quad 0 \quad 3])$. [Answer: 0]
- For $\boldsymbol{u} = [1 \quad -5 \quad 3]$ and $\boldsymbol{v} = [-10 \quad -6 \quad 2]$, evaluate $\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|^2}$. [Answer: $\frac{13}{70}$]
- For $\boldsymbol{u} = [1 \quad -6 \quad 1]$ and $\boldsymbol{v} = [1 \quad 2 \quad 3]$, evaluate $\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$. [Answer: $\left[-\frac{4}{7} \quad -\frac{8}{7} \quad -\frac{12}{7}\right]$]
- For $\boldsymbol{u} = [8 \quad 3 \quad -2]$ and $\boldsymbol{a} = [-1 \quad -2 \quad 2]$, evaluate $\frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$. [Answer: $[2 \quad 4 \quad -4]$]
- For $\boldsymbol{u} = [1 \quad 12 \quad 7]$ and $\boldsymbol{v} = [3 \quad 0 \quad -4]$, evaluate $\boldsymbol{u} - \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$. [Answer: $[4 \quad 12 \quad 3]$]
- For $\boldsymbol{u} = [1 \quad -1 \quad 1]$ and $\boldsymbol{v} = [-1 \quad -2 \quad 2]$, evaluate $\|\boldsymbol{u}+\boldsymbol{v}\|$, $\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$, and $\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$.

  [Answer: $\|\boldsymbol{u}+\boldsymbol{v}\| = 3\sqrt{2}$, $\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{\sqrt{3}}{3}$, $\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \left[-\frac{1}{3} \quad -\frac{2}{3} \quad \frac{2}{3}\right]$]
- Evaluate and simplify $[2 \quad 2 \quad 0] \cdot [x-1 \quad y-1 \quad z-3] = 0$.

  [Answer: $x + y - 2 = 0$]
- Evaluate and simplify $[x+2 \quad y-1 \quad z-7] \cdot [2 \quad 3 \quad -1] = 0$.

  [Answer: $2x + 3y - z + 8 = 0$]
- Evaluate and simplify $[-2 \quad 3 \quad 5] \cdot [x-7 \quad y-6 \quad z+2] = 0$.

  [Answer: $2x - 3y - 5z - 6 = 0$]
- Evaluate and simplify $[12 \quad -8 \quad -2] \cdot [x-2 \quad y+2 \quad z-5] = 0$.

  [Answer: $6x - 4y - z - 15 = 0$]