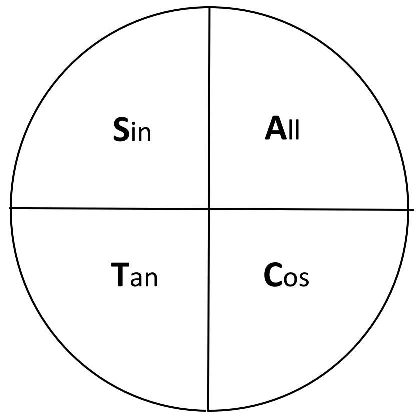
Trigonometric Functions and their Inverses

- All angles measured counter-clockwise from the positive x-axis are positive and the ones measured clockwise are negative.
- The CAST diagram reminds us which of the three basic trigonometric functions (sine, cosine, and tangent) are positive for standard position angle with terminal side in each of the four quadrants:



<u>Trigonometric functions of an arbitrary angle</u>:

For any trigonometric function f and any angle θ, $f(\theta) = \pm f(\phi)$, where ϕ is the reference angle defined by the acute angle between the terminal side of the standard position angle θ and the horizontal axis, and the sign matches the one in the CAST diagram.

	Domain	Range
sin	$\mathbb{R}$	$[-1,1]$
cos	$\mathbb{R}$	$[-1,1]$
tan	$\mathbb{R}\backslash\left\{\cdots,-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\cdots\right\}$	$(-\infty,\infty)$
csc	$\mathbb{R}\backslash\{\cdots,-2\pi,-\pi,0,\pi,2\pi,\cdots\}$	$(-\infty,-1]\cup[1,\infty)$
sec	$\mathbb{R}\backslash\left\{\cdots,-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\cdots\right\}$	$(-\infty,-1]\cup[1,\infty)$
cot	$\mathbb{R}\backslash\{\cdots,-2\pi,-\pi,0,\pi,2\pi,\cdots\}$	$(-\infty,\infty)$

sin, tan, csc, and cot are odd functions; cos and sec are even functions:

$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$	$\sec(-x) = \sec x$	$\cot(-x) = -\cot x$

Slope: $m = \frac{y_2-y_1}{x_2-x_1} = \tan\theta \quad (0° \leq \theta < 180°)$

Example Evaluate and simplify $\tan\left(-\frac{\pi}{4}\right)$ [Answer: -1]

Solution:

(The standard position angle of $-\frac{\pi}{4}$ has terminal side in the fourth quadrant, in which the tangent function is negative; moreover, its reference angle is $\phi = \frac{\pi}{4}$)

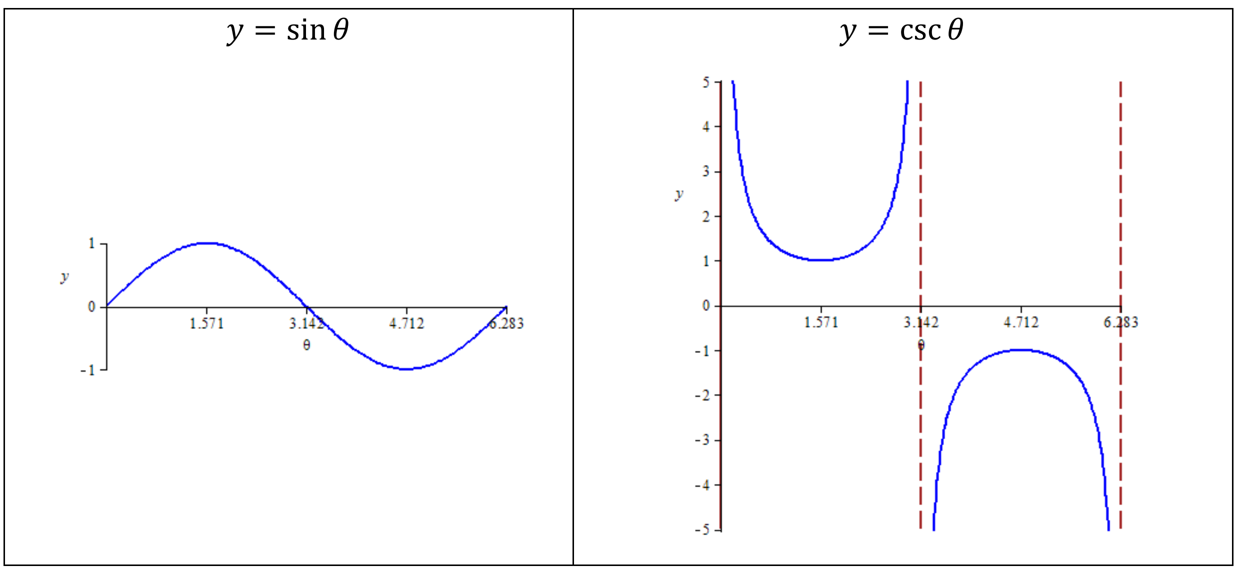
$$\tan\left(-\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -\frac{1}{1} = \mathbf{-1}$$

Exercise

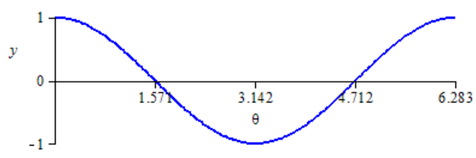
- Evaluate and simplify $\csc(780°)$. [Answer: $\frac{2\sqrt{3}}{3}$]
- Given that $\tan\theta = \frac{2}{3}$ and θ is in the third quadrant, find the other trigonometric function values.

  [Answer: $\sin\theta = -\frac{2\sqrt{13}}{13}, \cos\theta = -\frac{3\sqrt{13}}{13}, \csc\theta = -\frac{\sqrt{13}}{2}, \sec\theta = -\frac{\sqrt{13}}{3}, \cot\theta = \frac{3}{2}$]

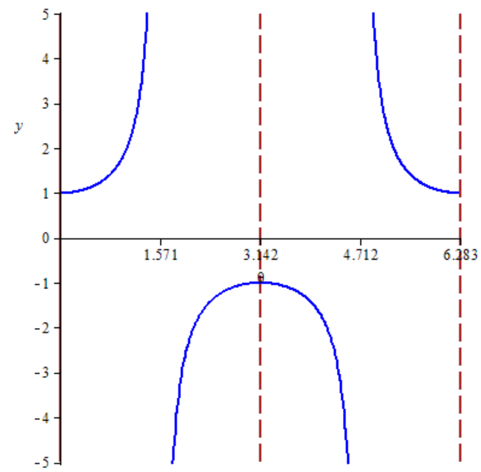
Graphs of the 6 trigononmetric functions (they are all periodic with period $2\pi \approx 6.283$)



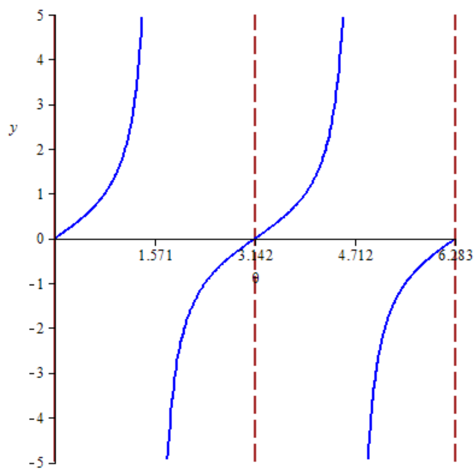
$y = \cos\theta$



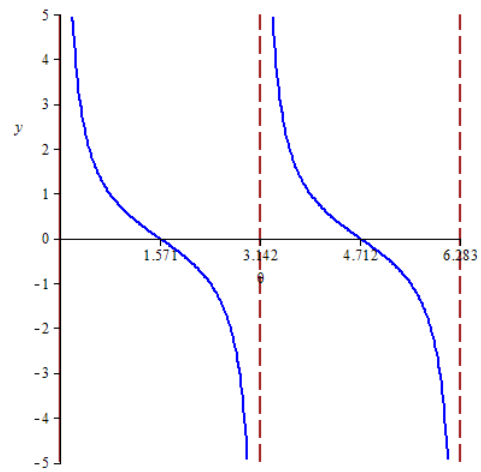
$y = \sec\theta$



$y = \tan\theta$



$y = \cot\theta$



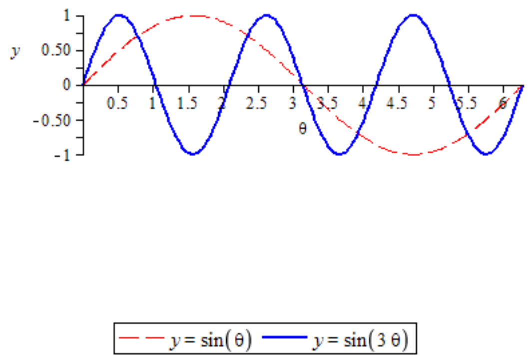
Graph of $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$:

- Amplitude $= |a|$
- Period $= \frac{2\pi}{b}$
- Displacement $= -\frac{c}{b}$

Example Sketch $f(\theta) = \sin(3\theta)$.

Solution:

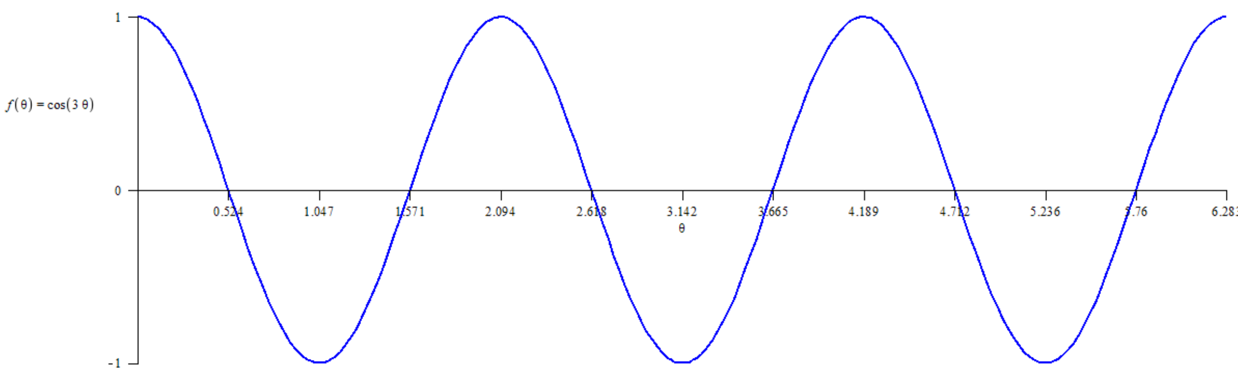
(Starting with the standard graph of $\sin(\theta)$, we get the graph of $\sin(3\theta)$ by horizontal compression with a factor of three; in other words, the period for $\sin(3\theta)$ is that of $\sin(\theta)$ divided by three, that is, $2\pi/3$; the transformed graph has θ-intercept at 0, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, π, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$, 2π, etc.)



Exercise

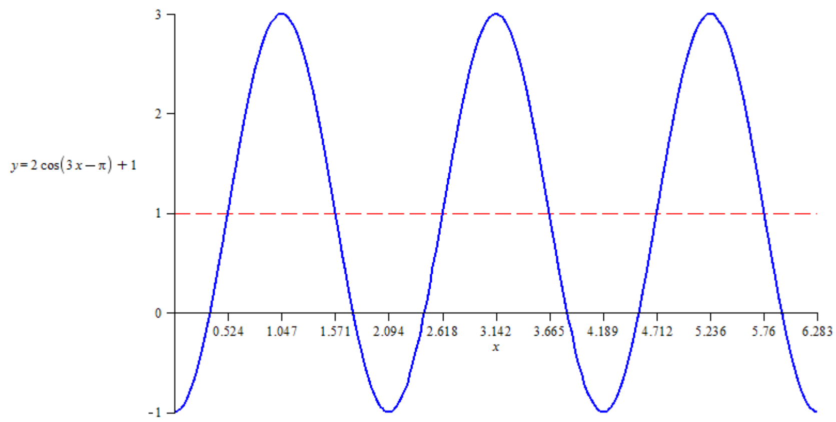
- Simplify $\frac{\csc(-\theta)}{\cot(-\theta)}$. [Answer: $\sec\theta$]
- Sketch $f(\theta) = \cos(3\theta)$.

  Answer:



- Sketch $y = 2\cos(3x - \pi) + 1$ by finding the amplitude, the period, and the phase shift.

  [Answer: Amplitude = 2, period = $\frac{2\pi}{3}$, phase shift = $\frac{\pi}{3}$]



Inverse Trigonometric Functions

	Domain	Range
$\sin^{-1}$	[-1, 1]	[-π/2, π/2]
$\cos^{-1}$	[-1, 1]	[0, π]
$\tan^{-1}$	(-∞,∞)	(-π/2, π/2)
$\csc^{-1}$	(-∞, 1] ∪ [1, ∞)	[-π/2, 0) ∪ (0, π/2]
$\sec^{-1}$	(-∞, 1] ∪ [1, ∞)	[0, π/2) ∪ (π/2, π]
$\cot^{-1}$	(-∞,∞)	(0, π)

Example Find the smallest positive integer N that solves the inequality $\frac{\pi}{2} - \tan^{-1} N \le 0.05$.

Solution:

$\frac{\pi}{2} - \tan^{-1} N \le 0.05 \Rightarrow \frac{\pi}{2} - 0.05 \le \tan^{-1} N$

(Apply the tangent function to both sides of the inequality; by definition of the inverse tangent function, $\tan^{-1} N < \frac{\pi}{2}$; since the tangent function is continuous and increasing on the interval $\left(0, \frac{\pi}{2}\right)$, the direction of the inequality is preserved; remember to set calculator in radian mode when computing $\tan\left(\frac{\pi}{2} - 0.05\right)$)

$\tan\left(\frac{\pi}{2} - 0.05\right) \le \tan(\tan^{-1} N) \Rightarrow \tan\left(\frac{\pi}{2} - 0.05\right) \le N \Rightarrow N \ge 19.983 \Rightarrow \boldsymbol{N \ge 20}$