Exercise Find the first four terms of the sequence $a_n = 3n - 1, n = 1, 2, 3, \cdots$.

[Answer: $a_1 = 2, a_2 = 5, a_3 = 8, a_4 = 11$]

Example Let $\{a_n\}$ be the sequence defined by $a_0 = 2$, $a_1 = 1$, and $(k+2)(k+1)a_{k+2} + (k+1)(k+3)a_{k+1} = 2a_k$ for $k = 0, 1, 2, \cdots$, find a_2 and a_3.

Solution:

$$(k=0)\quad (2)(1)a_2 + (1)(3)a_1 = 2a_0 \Longrightarrow 2a_2 = 2a_0 - 3a_1 \Longrightarrow a_2 = \frac{2(2)-3(1)}{2} \Longrightarrow \boldsymbol{a_2 = \frac{1}{2}}$$

$$(k=1)\quad (3)(2)a_3 + (2)(4)a_2 = 2a_1 \Longrightarrow 6a_3 = 2a_1 - 8a_2$$

$$\Longrightarrow a_3 = \frac{2(1)-8(1/2)}{6} = \frac{2-4}{6} \Longrightarrow \boldsymbol{a_3 = -\frac{1}{3}}$$

Exercise Without using a calculator, find the first 4 terms of the recursively defined sequence

- $a_1 = 4, a_{n+1} = 1 + \frac{1}{a_n}$ [Answer: $a_1 = 4, a_2 = \frac{5}{4}, a_3 = \frac{9}{5}, a_4 = \frac{14}{9}$]
- $a_1 = 2, a_2 = 3, a_{n+1} = a_n + a_{n-1}$ [Answer: $a_1 = 2, a_2 = 3, a_3 = 5, a_4 = 8$]

Example Evaluate $\sum_{n=0}^{4} \frac{(-1)^n}{n!(2n+1)}(0.5)^{2n+1}$ and round your answer to 5 decimal places.

Solution:

$$\sum_{n=0}^{4} \frac{(-1)^n}{n!(2n+1)}(0.5)^{2n+1}$$

$$= \frac{(-1)^0}{0!(2(0)+1)}(0.5)^{2(0)+1} + \frac{(-1)^1}{1!(2(1)+1)}(0.5)^{2(1)+1} + \frac{(-1)^2}{2!(2(2)+1)}(0.5)^{2(2)+1} +$$

$$+ \frac{(-1)^3}{3!(2(3)+1)}(0.5)^{2(3)+1} + \frac{(-1)^4}{4!(2(4)+1)}(0.5)^{2(4)+1}$$

$$= \frac{1}{(1)(1)}(0.5)^1 + \frac{(-1)}{(1)(3)}(0.5)^3 + \frac{1}{(2)(5)}(0.5)^5 + \frac{(-1)}{(6)(7)}(0.5)^7 + \frac{1}{(24)(9)}(0.5)^9$$

$$= 0.5 - 0.041666 + 0.003125 - 0.000186 + 0.000009 \approx \mathbf{0.46128}$$

Exercise Find and evaluate each of the following sums

- $\sum_{n=1}^{5} n^3$ [Answer: 225]

Arithmetic Sequences

$a_n = a_{n-1} + d$ and $a_n = a_1 + (n-1)d$

$S_n = \frac{n}{2}(a_1 + a_n)$ \

Exercise

- For the arithmetic sequence $4, 7, 10, 13, \cdots$
  (a) find the 2011^{th} term; [Answer: 6,034]
  (b) which term is 295? [Answer: 98^{th} term]
  (c) find the sum of the first 750 terms. [Answer: 845,625]
- Find the sum of the first 800 natural numbers. [Answer: 320,400]

Geometric Sequences

$a_n = ra_{n-1}$ and $a_n = a_1 r^{n-1}$

$S_n = \frac{a_1(1-r^n)}{1-r}$

Infinite Geometric Series: For $|r| < 1, S = \frac{a_1}{1-r}$

Example Find the exact value of the geometric series $\frac{10}{9} + \frac{50}{81} + \frac{250}{729} + \cdots$.

Solution:

(Check that the given series is a geometric series by evaluating ratio of consecutive terms)

$\frac{50/81}{10/9} = \frac{50}{81} \times \frac{9}{10} = \frac{50}{10} \times \frac{9}{81} = 5 \times \frac{1}{9} = \frac{5}{9}$;

$\frac{250/729}{50/81} = \frac{250}{729} \times \frac{81}{50} = \frac{250}{50} \times \frac{81}{729} = 5 \times \frac{1}{9} = \frac{5}{9}$;

geometric series with common ratio $r = \frac{5}{9}$

(Since $|r| = \frac{5}{9} < 1$, apply the infinite sum formula for geometric series)

$$S = \frac{a_1}{1-r} = \frac{10/9}{1-(5/9)} = \frac{10/9}{4/9} = \frac{10}{9} \times \frac{9}{4} = \frac{10}{4} \times \frac{9}{9} = \mathbf{\frac{5}{2}}$$

Exercise

- Find the sum of the first 10 terms of the geometric sequence $3, 15, 75, 375, \cdots$.
  [Answer: 7,324,218]
- Find the value of the infinite series $\sum_{n=0}^{\infty} \frac{2}{5^n}$. [Answer: $\frac{5}{2}$]

Maclaurin Series

- $f(x)=f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\cdots+\frac{f^{(n)}(0)x^n}{n!}+\cdots$
- $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ (all x)
- $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$ (all x)
- $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$ (all x)
- $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ (for $|x|<1$)
- (Binomial Series) $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots$ $(|x|<1)$

Example Find the first three non-zero terms of the Maclaurin series for $\frac{1}{\sqrt[4]{1+x^2}}$.

Solution:

(Method 1: Use the definition – straight forward but tedious, in comparison with method 2)

$f(x)=(1+x^2)^{-1/4}\Rightarrow f(0)=(1+0)^{-1/4}=(1)^{-1/4}=1$ (first non − zero term);

$f'(x)=-\frac{1}{4}(1+x^2)^{-5/4}\cdot(2x)=-\frac{x}{2}(1+x^2)^{-5/4}$

$\Rightarrow f'(0)=-\frac{0}{2}(1+0^2)^{-5/4}=0$ (this zero term doesn't count);

$f''(x)=-\frac{1}{2}(1+x^2)^{-5/4}-\frac{x}{2}\cdot\left(-\frac{5}{4}\right)(1+x^2)^{-9/4}\cdot(2x)$

$=-\frac{1}{2}(1+x^2)^{-5/4}+\frac{5x^2}{4}(1+x^2)^{-9/4}=\frac{(1+x^2)^{-9/4}}{4}[-2(1+x^2)+5x^2]$

$=\frac{(1+x^2)^{-9/4}}{4}[-2-2x^2+5x^2]=\frac{(1+x^2)^{-9/4}}{4}[3x^2-2]=\frac{1}{4}(3x^2-2)(1+x^2)^{-9/4}$

$\Rightarrow f''(0)=\frac{1}{4}(3\cdot 0^2-2)(1+0^2)^{-9/4}=\frac{1}{4}(-2)(1)=-\frac{1}{2}$ (second non − zero term)

$f'''(x)=\frac{1}{4}\left[(6x)(1+x^2)^{-9/4}+(3x^2-2)\cdot\left(-\frac{9}{4}\right)(1+x^2)^{-13/4}\cdot(2x)\right]$

$=\frac{1}{4}\left[(6x)(1+x^2)^{-9/4}-\frac{9x}{2}(3x^2-2)(1+x^2)^{-13/4}\right]$

$=\frac{1}{4}\left(\frac{3x}{2}\right)(1+x^2)^{-13/4}[4(1+x^2)-3(3x^2-2)]$

$=\frac{3x}{8}(1+x^2)^{-13/4}[4+4x^2-9x^2+6]=\frac{3x}{8}(1+x^2)^{-13/4}(10-5x^2)$

$=\frac{15x}{8}(1+x^2)^{-13/4}(2-x^2)$

$\Rightarrow f'''(x)=\frac{15(0)}{8}(1+0^2)^{-13/4}(2-0^2)=0$ (this zero term doesn't count);

$f^{(4)}(x)=\frac{15}{8}\frac{d}{dx}\left[(2x-x^3)(1+x^2)^{-13/4}\right]$

$=\frac{15}{8}\left[(2-3x^2)(1+x^2)^{-13/4}+(2x-x^3)\cdot\left(-\frac{13}{4}\right)(1+x^2)^{-17/4}\cdot(2x)\right]$

$=\frac{15}{8}\left[(2-3x^2)(1+x^2)^{-13/4}-\frac{13x}{2}(2x-x^3)(1+x^2)^{-17/4}\right]$

$\Rightarrow f^{(4)}(0) = \frac{15}{8}\left[(2-3\cdot 0^2)(1+0^2)^{-13/4} - \frac{13\cdot 0}{2}(2\cdot 0 - 0^3)(1+0^2)^{-17/4}\right]$

$=\frac{15}{4}$ (third non − zero term)

Hence $\frac{1}{\sqrt[4]{1+x^2}} = 1 + \frac{1}{2!}\left(-\frac{1}{2}\right)x^2 + \frac{1}{4!}\left(\frac{15}{4}\right)x^4 + \cdots$

$= 1 + \frac{1}{2}\left(-\frac{1}{2}\right)x^2 + \frac{1}{24}\left(\frac{15}{4}\right)x^4 + \cdots = \mathbf{1 - \frac{1}{4}x^2 + \frac{5}{32}x^4} + \cdots$

(Method 2: Use the binomial series and replace x and n by x^2 and $-\frac{1}{4}$ respectively)

$\frac{1}{\sqrt[4]{1+x^2}} = (1+x^2)^{-1/4} = 1 + \left(-\frac{1}{4}\right)(x^2) + \frac{1}{2}\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)(x^2)^2 + \cdots$

$= 1 - \frac{1}{4}x^2 + +\frac{1}{2}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)x^4 + \cdots = \mathbf{1 - \frac{1}{4}x^2 + \frac{5}{32}x^4} + \cdots$

Example Find the power series of $\tan^{-1} x$ centered at zero.

Solution:

(Since $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$, we can find the power series of $\tan^{-1} x$ by integrating the power series of $\frac{1}{1+x^2}$, which can be obtained by using the Binomial series)

$\frac{1}{1+x^2} = (1+x^2)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)(-2)\cdots(-n+1)(-n)}{n!}(x^2)^n$

$= \sum_{n=0}^{\infty} \frac{(-1)^n n(n-1)\cdots(2)(1)}{n!}x^{2n} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

$\Rightarrow \tan^{-1} x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$

$= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C$

(To determine C, evaluate both sides at a convenient value of x, say $x = 0$)

$\tan^{-1} 0 = \sum_{n=0}^{\infty}(-1)^n \frac{0^{2n+1}}{2n+1} + C \Rightarrow 0 = 0 + C \Rightarrow C = 0,$ hence

$\boldsymbol{\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}}$

Exercise

- By direct expansion, find the first four nonzero terms of the Maclaurin expansion for $f(x) = (1+e^x)^2$. [Answer: $(1+e^x)^2 = 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \cdots$]
- Find the first three nonzero terms of the expansion for $f(x) = \frac{1}{\sqrt{1-2x}}$ by using the binomial series. [Answer: $\frac{1}{\sqrt{1-2x}} = 1 + x + \frac{3}{2}x^2 + \cdots$]

Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots +$$

Example Approximate $\sqrt{9.6}$ by using the second Taylor polynomial of $f(x) = \sqrt{x}$ centered at $a = 9$.

Solution:

$$f(x) = \sqrt{x} \quad \Rightarrow \quad f(a) = \sqrt{9} = 3;$$
$$f'(x) = \frac{1}{2}x^{-1/2} \quad \Rightarrow \quad f'(a) = \frac{1}{2\sqrt{9}} = \frac{1}{6};$$
$$f''(x) = -\frac{1}{4}x^{-3/2} \quad \Rightarrow \quad f''(a) = -\frac{1}{4\left(\sqrt{9}^3\right)} = -\frac{1}{108};$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\sqrt{9.6} = f(9.6) \approx P_2(9.6) = 3 + \frac{1}{6}(9.6-9) + \frac{(-1/108)}{2!}(9.6-9)^2$$

$$= 3 + \frac{1}{6}(0.6) - \frac{1}{216}(0.36) \approx \mathbf{3.09833}$$

Exercise

- Find the first three nonzero terms of the Taylor expansion for $f(x) = \cos x$, with $a = \frac{\pi}{3}$.

  [Answer: $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \cdots$]
- Expand $f(x) = e^x$ in a Taylor series centered at $a = 1$.

  [Answer: $e^x = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \cdots$]
- Expand $f(x) = \sqrt{x}$ in powers of $(x-4)$.

  [Answer: $\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \cdots$]