Rational Expressions

Properties of Rational Expressions

- Domain of a Rational Expression: Denominator is non-zero
- Reducing to Lowest Terms/Building Up the Denominator:

  Basic Principle of Rational Numbers: $\dfrac{ac}{bc} = \dfrac{a}{b}$

Multiplication & Division of Rational Expressions

- Multiplication: $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$
- Division: $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c}$

Example Simplify $\frac{x^2-2x-35}{2x^3-3x^2} \cdot \frac{4x^3-9x}{6x-42}$.

Solution:

$$\frac{x^2-2x-35}{2x^3-3x^2} \cdot \frac{4x^3-9x}{6x-42} = \frac{(x-7)(x+5)}{x^2(2x-3)} \cdot \frac{x(4x^2-9)}{6(x-7)} = \frac{(x-7)(x+5)}{x^2(2x-3)} \cdot \frac{x(2x+3)(2x-3)}{6(x-7)} = \boldsymbol{\frac{(x+5)(2x+3)}{6x}}$$

Exercise Simplify $\frac{x+4}{x-3} \div \frac{x^2-x-2}{x^2-9}$. [Answer: $\frac{(x+4)(x+3)}{(x-2)(x+1)}$]

Exercise Simplify $\frac{t^3-1}{t^2-1} \div \frac{t^2+t+1}{t^2+2t+1}$. [Answer: $t+1$]

Addition & Subtraction of Rational Expressions

Build up each rational expression to equivalent forms with identical denominators (preferably the LCD – Least Common Denominator)

Rational Expressions

Example Simplify $\frac{9x+2}{3x^2-2x-8}+\frac{7}{3x^2+x-4}$.

Solution:

$$\frac{9x+2}{3x^2-2x-8}+\frac{7}{3x^2+x-4}=\frac{9x+2}{(x-2)(3x+4)}+\frac{7}{(x-1)(3x+4)}=\frac{9x+2}{(x-2)(3x+4)}+\frac{7}{(x-1)(3x+4)}$$

$$=\frac{(x-1)(9x+2)}{(x-1)(x-2)(3x+4)}+\frac{7(x-2)}{(x-1)(x-2)(3x+4)}=\frac{(9x^2+2x-9x-2)+(7x-14)}{(x-1)(x-2)(3x+4)}=\frac{9x^2-16}{(x-1)(x-2)(3x+4)}$$

$$=\frac{(3x+4)(3x-4)}{(x-1)(x-2)(3x+4)}=\frac{\mathbf{3x-4}}{\mathbf{(x-1)(x-2)}}$$

Complex Fractions

Simplifying Complex Fraction: Multiply the numerator and denominator by the LCD for all of the fractions in the complex fraction

Example Simplify $\frac{\frac{t}{1-t}+\frac{1+t}{t}}{\frac{1-t}{t}+\frac{t}{1+t}}$.

Solution:

$$\frac{\frac{t}{1-t}+\frac{1+t}{t}}{\frac{1-t}{t}+\frac{t}{1+t}}=\frac{\left(\frac{t}{1-t}+\frac{1+t}{t}\right)[t(1-t)(1+t)]}{\left(\frac{1-t}{t}+\frac{t}{1+t}\right)[t(1-t)(1+t)]}=\frac{t^2(1+t)+(1-t)(1+t)^2}{(1-t)^2(1+t)+t^2(1-t)}=\frac{(1+t)[t^2+(1-t)(1+t)]}{(1-t)[(1-t)(1+t)+t^2]}$$

$$=\frac{(1+t)[t^2+1-t^2]}{(1-t)[1-t^2+t^2]}=\frac{\mathbf{(1+t)}}{\mathbf{(1-t)}}$$

Exercise Simplify $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^3}+\frac{1}{y^3}}$. [Answer: $\frac{x^2y^2}{x^2-xy+y^2}$]

Solving Equations involving Rational Expressions

- Multiply each side of the equation by the LCD of the rational expressions to eliminate all denominators
- Beware of extraneous roots: check all answers
- Using the Extremes - Means Property to solve a Proportion : $\frac{a}{b}=\frac{c}{d}\Rightarrow ad=bc$

Rational Expressions

Example Solve the rational equation $\frac{3}{x+1} - \frac{2}{x-1} = \frac{2}{(x-2)(x-1)}$.

Solution:

(Clear all the denominators by multiplying both sides of the equation with

$(x+1)(x-1)(x-2)$, the least common multiple of all the denominators)

$$(x+1)(x-1)(x-2)\left[\frac{3}{x+1} - \frac{2}{x-1}\right] = \frac{2}{(x-2)(x-1)} \cdot (x+1)(x-1)(x-2)$$

$$\Longrightarrow 3(x-1)(x-2) - 2(x+1)(x-2) = 2(x+1)$$

(Expand both sides, bring all the terms to one side and simplify by collecting like terms)

$$\Longrightarrow 3(x^2 - 3x + 2) - 2(x^2 - x - 2) - 2(x+1) = 0$$

$$\Rightarrow 3x^2 - 9x + 6 - 2x^2 + 2x + 4 - 2x - 2 = 0 \Longrightarrow x^2 - 9x + 8 = 0$$

(Solve the resulting equation, by factoring the left hand side in this example) $\Longrightarrow$

$$(x-8)(x-1) = 0 \Rightarrow x = 8 \text{ or } x = 1$$

(Check that the answers found satisfy the original equation)

$$x = 8:\ \frac{3}{8+1} - \frac{2}{8-1} = \frac{2}{(8-2)(8-1)}\ ?\ \Leftrightarrow\ \frac{3}{9} - \frac{2}{7} = \frac{2}{(6)(7)}\ ?\ \Leftrightarrow\ \frac{1}{3} - \frac{2}{7} = \frac{1}{21}\ ?\quad Yes!$$

$$x = 1:\ \frac{3}{1+1} - \frac{2}{1-1} = \frac{2}{(1-2)(1-1)}\ ?\ \Leftrightarrow\ \frac{3}{2} - \frac{2}{0} = \frac{2}{(-1)(0)}\ ?\quad No!$$

Hence $\boldsymbol{x = 8}$.

Exercise Solve the rational equation.

- $\frac{x-9}{3} + \frac{x-4}{2} = 0.$ [Answer: $x = 6$]

Example For what positive values of x is the function $f(x) = \frac{1}{40+12x-4x^2}$ positive?

Solution:

(Find all the zeros of the denominator) $40 + 12x - 4x^2 = 0 \Longrightarrow 4(10 + 3x - x^2) = 0$

(Factor the quadratic factor using the ac-method):

$$4(10 + 3x - x^2) = 0 \Longrightarrow 4(10 + 5x - 2x - x^2) = 0$$

$$\Longrightarrow 4[(10 + 5x) + (-2x - x^2)] = 0 \Longrightarrow 4[5(2+x) - x(2+x)] = 0$$

$$\Longrightarrow 4(2+x)[5-x] = 0 \Longrightarrow x = -2 \quad \text{or} \quad x = 5$$

Rational Expressions

(Use the zeros found to divide the real number line into intervals)

$$x < -2, \quad -2 < x < 5, \quad x > 5$$

(Determine which of the intervals we should include in the solution, that is, satisfy the inequality we try to solve, by using test points)

Interval	$(-\infty, -2)$	$(-2, 5)$	$(5, \infty)$
Test Point (one of many possibilities)	–3	0	6
Inequality $\frac{1}{4(2+x)(5-x)} > 0$ Satisfied?	$\frac{(+)}{(+)(-)(+)} > 0?$	$\frac{(+)}{(+)(+)(+)} > 0?$	$\frac{(+)}{(+)(+)(-)} > 0?$
Part of the Solution?	No	Yes	No

(None of the zeros satisfy the inequality, and we determine the solution)

$$\mathbf{-2} < x < 5 \quad \text{or} \quad (-2,5)$$