# Quadratic Equations

Solving a Quadratic Equation $(ax^2 + bx + c = 0)$

- Factoring (Zero Product Property: $a \times b = 0 \Rightarrow a = 0$ or $b = 0$)
- Completing the Square:

$$ax^2 + bx + c = 0 \Rightarrow ax^2 + bx = -c \Rightarrow a\left[x^2 + \frac{b}{a}x\right] = -c$$

$$\Rightarrow a\left[\left\{x^2 + \frac{b}{a}x + \left(\frac{b/a}{2}\right)^2\right\} - \left(\frac{b/a}{2}\right)^2\right] = -c \Rightarrow a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] = -c$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a} \Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Quadratic Formula: $ax^2 + bx + c = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Using the determinant $D = b^2 - 4ac$ to decide the number and the nature of the solutions of a quadratic equation

- $D > 0$ is a perfect square: the roots are real, rational, and unequal
- $D > 0$ is not a perfect square: the roots are real, irrational, and unequal
- $D = 0$: the roots are real, rational, and equal
- $D < 0$: the roots are complex numbers and unequal (but conjugates)

Example Solve $x^2 + x = 6$

Solution:

$x^2 + x = 6 \quad \Rightarrow \quad x^2 + x - 6 = 0$

(Since the discriminant $D = b^2 - 4ac = 1^2 - 4(1)(-6) = 1 + 24 = 25$ is the square of the rational number 5, an easy way to solve this quadratic equation is by factoring the quadratic expression, using the $ac$-method)

$$x^2 + x - 6 = 0 \Rightarrow x^2 + 3x - 2x - 6 = 0 \Rightarrow (x^2 + 3x) + (-2x - 6) = 0$$

$$\Rightarrow x(x + 3) - 2(x + 3) = 0 \Rightarrow (x + 3)(x - 2) = 0 \Rightarrow \boldsymbol{x = -3 \text{ or } x = 2}$$

Exercise Solve the following equations.

- $x^2 - 4x = 0$ [Answer: $x = 0, 4$]
- $t^2 + 2t = 0$ [Answer: $t = -2, 0$]
- $3x^2 + 12x = 0$ [Answer: $x = -4, 0$]
- $100 - m^2 = 0$ [Answer: $m = -10, 10$]
- $m^2 = 3 - 2m$ [Answer: $m = -3, 1$]
- $s^2 - 8s = 9$ [Answer: $s = -1, 9$]
- $8y^2 + 24 = 32y$ [Answer: $y = 1, 3$]
- $x^2 - 8 = 2x$ [Answer: $x = -2, 4$]
- $2u^2 = 3 + u$ [Answer: $u = -1, \frac{3}{2}$]
- $k^2 - 4 = 0$ [Answer: $k = -2, 2$]
- $2t^2 - 10 = 0$ [Answer: $t = \sqrt{5}, -\sqrt{5}$]
- $x^2 - 6x - 10 = 0$ [Answer: $x = 3 - \sqrt{19}, 3 + \sqrt{19} \approx -1.359, 7.359$]
- $2y^2 - 1 = 3y$ [Answer: $y = \frac{3-\sqrt{17}}{4}, \frac{3+\sqrt{17}}{4} \approx -0.281, 1.781$]

Exercise Solve $r^2 + z^2 = 36$ for $z$. [Answer: $z = \pm\sqrt{36 - r^2}$]

Example Solve $t^2 - 6t + 9 = 0$

Solution:

(Since the discriminant $D = b^2 - 4ac = (-6)^2 - 4(1)(9) = 36 - 36 = 0$, the quadratic equation has equal real rational roots; an easy way to solve this quadratic equation is by factoring the quadratic expression as perfect square trinomial)

$t^2 - 6t + 9 = 0 \Rightarrow t^2 - 2(3)(t) + (3)^2 = 0 \Rightarrow (t-3)^2 = 0 \Rightarrow \boldsymbol{t = 3 \text{ or } t = 3}$

Exercise Solve the following equation.

- $r^2 - 10r + 25 = 0$ [Answer: $r = 5, 5$]

Example Solve $m^2 + 2m + 5 = 0$

Solution:

(Since $b^2 - 4ac = 2^2 - 4(1)(5) = 4 - 20 = -16$ is not the square of a rational number, we cannot factor the left-hand-side and hence the easiest way to solve this quadratic equation is by applying the quadratic formula)

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{2^2-4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = \mathbf{-1 \pm 2i}$$

Exercise Solve the following equations.

- $s^2 + 9 = 0$ [Answer: $s = -3i, 3i$]

Example Solve the following equations. $t^3 - 2t = 0$

Solution:

(Factor out common factor) $t(t^2 - 2) = 0$

(Use the zero product property: $ab = 0 \Longrightarrow a = 0$ or $b = 0$)

$$\Rightarrow t = 0 \text{ or } t^2 - 2 = 0 \Rightarrow t = 0 \text{ or } t = \frac{-0 \pm \sqrt{0^2 - 4(1)(-2)}}{2(1)}$$

$$\Longrightarrow t = 0 \text{ or } t = \pm\frac{\sqrt{8}}{2} \Longrightarrow t = 0 \text{ or } t = \pm\frac{2\sqrt{2}}{2} \Rightarrow \mathbf{t = 0 \text{ or } t = \sqrt{2} \text{ or } t = -\sqrt{2}}$$

Solving equations quadratic in form

Example $x^4 - 5x^2 + 4 = 0$. [Answer: $x = -2, -1, 1, 2$]

Solution:

(Rewrite the equation as a quadratic equation by treating $x^2$ as a new variable)

$(x^2)^2 - 5(x^2) + 4 = 0$

(Since the discriminant $D = b^2 - 4ac = (-5)^2 - 4(1)(4) = 25 - 16 = 9$ is a perfect square, we can solve the equation by factoring the quadratic expression using either the $ac$-method or trial and error) $\Rightarrow (x^2)^2 - (x^2) - 4(x^2) + 4 = 0$

$\Rightarrow [(x^2)^2 - (x^2)] + [-4(x^2) + 4] = 0 \Rightarrow (x^2)[(x^2) - 1] - 4[(x^2) - 1] = 0$

$\Rightarrow [(x^2) - 1][(x^2) - 4] = 0$

(Use the factoring formula $a^2 - b^2 = (a + b)(a - b)$ to factor the left hand side completely) $\Rightarrow (x^2 - 1^2)(x^2 - 2^2) = 0 \Rightarrow (x + 1)(x - 1)(x + 2)(x - 2) = 0$

(Solve for $x$ and check the answers in the original equation) $\mathbf{x = -1, 1, -2, 2}$

$x = -1$: $(-1)^4 - 5(-1)^2 + 4 = 0\,?$ $\Leftrightarrow$ $1 - 5 + 4 = 0\,?$ $Yes!$

$x = 1$: $(1)^4 - 5(1)^2 + 4 = 0\,?$ $\Leftrightarrow$ $1 - 5 + 4 = 0\,?$ $Yes!$

$x = -2$: $(-2)^4 - 5(-2)^2 + 4 = 0\ ? \quad \Leftrightarrow \quad 16 - 20 + 4 = 0\ ? \quad Yes!$

$x = 1$: $(2)^4 - 5(2)^2 + 4 = 0\ ? \quad \Leftrightarrow \quad 16 - 20 + 4 = 0\ ? \quad Yes!$