

# Differentiation

## Applications of Derivatives: Tangents

$$\Delta x = x_2 - x_1; \quad \Delta y = y_2 - y_1 = f(x_1 + \Delta x) - f(x_1)$$

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

$$m_{\text{tan}} = \lim_{\Delta x \rightarrow 0} m_{PQ} = \lim_{\Delta x \rightarrow 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

## Definition of Derivative: Delta Process

$$\Delta y = (y + \Delta y) - y = f(x + \Delta x) - f(x)$$

$$f'(x) \left( \text{or } \frac{dy}{dx} \right) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example Find the derivative of  $y = 2x^3 + \frac{4}{x}$  by using the definition.

**Solution:**

$$f(x) = 2x^3 + \frac{4}{x}$$

$$\Rightarrow f(x + \Delta x) = 2(x + \Delta x)^3 + \frac{4}{x + \Delta x} = 2[x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3] + \frac{4}{x + \Delta x}$$

$$\Rightarrow f(x + \Delta x) - f(x) = 2[x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3] + \frac{4}{x + \Delta x} - 2x^3 - \frac{4}{x}$$

$$= 2x^3 + 6x^2(\Delta x) + 6x(\Delta x)^2 + 2(\Delta x)^3 - 2x^3 + \frac{4}{x + \Delta x} - \frac{4}{x}$$

$$= 6x^2(\Delta x) + 6x(\Delta x)^2 + 2(\Delta x)^3 + \frac{4x - 4(x + \Delta x)}{x(x + \Delta x)}$$

$$= 6x^2(\Delta x) + 6x(\Delta x)^2 + 2(\Delta x)^3 + \frac{4x - 4x - 4(\Delta x)}{x(x + \Delta x)}$$

$$= 6x^2(\Delta x) + 6x(\Delta x)^2 + 2(\Delta x)^3 - \frac{4(\Delta x)}{x(x + \Delta x)}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \left[ \frac{6x^2(\Delta x)}{(\Delta x)} + \frac{6x(\Delta x)^2}{(\Delta x)} + \frac{2(\Delta x)^3}{(\Delta x)} - \frac{4(\Delta x)}{x(x + \Delta x)(\Delta x)} \right]$$

$$= \lim_{\Delta x \rightarrow 0} \left[ 6x^2 + 6x(\Delta x) + 2(\Delta x)^2 - \frac{4}{x(x + \Delta x)} \right]$$

$$= \lim_{\Delta x \rightarrow 0} (6x^2) + \lim_{\Delta x \rightarrow 0} [6x(\Delta x)] + \lim_{\Delta x \rightarrow 0} [2(\Delta x)^2] - \lim_{\Delta x \rightarrow 0} \frac{4}{x(x + \Delta x)}$$

$$= 6x^2 + [6x(0)] + [2(0)^2] - \frac{4}{x(x+0)} = 6x^2 - \frac{4}{x^2}$$

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Exercise Find the derivative  $dy/dx$  by using the definition.

- $y = 3x^2 + 2x$  [Answer:  $\frac{dy}{dx} = 6x + 2$ ]
- $y = 5x - 2x^2$  [Answer:  $\frac{dy}{dx} = 5 - 4x$ ]
- $y = \frac{3}{x+2}$  [Answer:  $\frac{dy}{dx} = -\frac{3}{(x+2)^2}$ ]
- $y = \sqrt{x}$  [Answer:  $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ ]

### Basic Differentiation Formulas

- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(cu) = c \frac{d}{dx}(u)$
- $\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$

Chain Rule:  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

### Derivative of Polynomials

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dx}(ax^2 - x^2 - 4ax + 4x + 5a)$  [Answer:  $2ax - 2x - 4a + 4$ ]
- $\frac{d}{dx}(11x^2 + 14ax + 4a^2 - 48x - 36a + 5)$  [Answer:  $22x + 14a - 48$ ]
- $\frac{d}{dx}\left(\frac{2+3x}{5}\right)$  [Answer:  $\frac{3}{5}$ ]
- $\frac{d}{dx}(4x^6 - 2x^4 + \pi^3)$  [Answer:  $24x^5 - 8x^3$ ]

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**Example**  $\frac{d}{dx} \sum_{n=0}^N (-1)^n x^n$

**Solution:**

(Derivative of a sum is the sum of derivatives)

$$\frac{d}{dx} \sum_{n=0}^N (-1)^n x^n = \sum_{n=0}^N \frac{d}{dx} [(-1)^n x^n] = \sum_{n=0}^N (-1)^n n x^{n-1}$$

(Note that the coefficient  $(-1)^n n$  is zero when  $n = 0$ , hence we can start the summation with  $n = 1$ )

$$= \sum_{n=1}^N (-1)^n n x^{n-1} = (-1)^1 1 x^0 + (-1)^2 2 x^1 + (-1)^3 3 x^2 + \dots + (-1)^N N x^{N-1}$$

(observe that this can be written as a finite series involving  $x^n$ )

$$= \sum_{n=0}^{N-1} (-1)^{n+1} (n+1) x^n$$

(This is an example of applying the technique of index-shifting: we increase the running index by one in the summand – replace  $n$  by  $n + 1$ , and decrease the value of the running index in the summation notation by one –  $n$  runs from 0 to  $N - 1$  instead.)

**Exercise** Evaluate and simplify the following derivatives.

- $\frac{d}{dx} \sum_{i=1}^n [y_i - a - x t_i^3]^2$  [Answer:  $-2(\sum_{i=1}^n t_i^3 y_i - a \sum_{i=1}^n t_i^3 - x \sum_{i=1}^n t_i^6)$ ]

- For  $y = \sum_{N=0}^{\infty} a_N x^N$ , evaluate and simplify  $x \frac{d^2 y}{dx^2}$ .

[Answer:  $\sum_{N=2}^{\infty} N(N-1) a_N x^{N-1} = \sum_{N=1}^{\infty} (N+1) N a_{N+1} x^N$ ]

### Derivative of Trigonometric Functions

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx} \sin x = \cos x$	$\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$
$\frac{d}{dx} \cos x = -\sin x$	$\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$
$\frac{d}{dx} \tan x = \sec^2 x$	$\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$
$\frac{d}{dx} \csc x = -\csc x \cot x$	$\frac{d}{dx} \csc u = -\csc u \cot u \frac{du}{dx}$
$\frac{d}{dx} \sec x = \sec x \tan x$	$\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx}$
$\frac{d}{dx} \cot x = -\csc^2 x$	$\frac{d}{dx} \cot u = -\csc^2 u \frac{du}{dx}$

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Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dt} (t - \sin t)$  [Answer:  $1 - \cos t$ ]
- $\frac{d}{dt} (\sec t - 1)$  [Answer:  $\sec t \tan t$ ]
- $\frac{d}{dt} \sin(2t)$  [Answer:  $2 \cos(2t)$ ]
- $\frac{d}{dr} \sin(3r^2)$  [Answer:  $6r \cos(3r^2)$ ]

### Derivative of Inverse Trigonometric Functions

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$
$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx} \cos^{-1} u = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$
$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$	$\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dx} [\tan^3(2x) + \tan^{-1}(2x)]$  [Answer:  $6 \tan^2(2x) \sec^2(2x) + \frac{2}{1+4x^2}$ ]

### Derivative of Exponential and Logarithmic Functions

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx} b^x = b^x \ln b$	$\frac{d}{dx} b^u = b^u \ln b \frac{du}{dx}$
$\frac{d}{dx} e^x = e^x$	$\frac{d}{dx} e^u = e^u \frac{du}{dx}$
$\frac{d}{dx} \log_b x = \frac{1}{x} \log_b e$	$\frac{d}{dx} \log_b u = \frac{1}{u} \log_b e \frac{du}{dx}$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dt} (a^2 e^t + 5b \cos c + a^3 \sin 3c - b^4 t - 12)$  [Answer:  $a^2 e^t - b^4$ ]
- $\frac{d}{dt} [\cos(8t) + e^{2t}]$  [Answer:  $-8 \sin(8t) + 2e^{2t}$ ]
- $\frac{d}{dy} \ln(3y^4)$  [Answer:  $\frac{4}{y}$ ]

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- $\frac{d}{du} \left( \frac{2}{3} \ln \frac{5-u}{2} \right)$  [Answer:  $\frac{2}{3(u-5)}$ ]
- $\frac{d}{dy} \ln(\cos e^{2y})$  [Answer:  $-2e^{2y} \tan e^{2y}$ ]
- $\frac{d}{du} \ln[(\sin 2u)\sqrt{u^2 + 1}]$  [Answer:  $2 \cot 2u + \frac{u}{u^2+1}$ ]
- $\frac{d}{dt} \ln \frac{t-1}{t+1}$  [Answer:  $\frac{2}{(t-1)(t+1)}$  or  $\frac{2}{t^2-1}$ ]

### Derivative of a Power of a Function

Basic Formula	Basic Formula with Chain Rule
$\frac{d}{dx} (x^n) = nx^{n-1}$	$\frac{d}{dx} (u^n) = nu^{n-1} \cdot \frac{d}{dx} (u)$
$\frac{d}{dx} (x^{p/q}) = \frac{p}{q} x^{(p/q)-1}$	$\frac{d}{dx} (u^{p/q}) = \frac{p}{q} u^{(p/q)-1} \cdot \frac{d}{dx} (u)$

Example Evaluate  $\frac{d}{dt} \ln^3(1 - 2t)$ .

**Solution:**

$$\begin{aligned} \frac{d}{dt} \ln^3(1 - 2t) &= \frac{d}{dt} \{[\ln(1 - 2t)]^3\} = 3[\ln(1 - 2t)]^2 \cdot \frac{d}{dt} [\ln(1 - 2t)] \\ &= 3[\ln(1 - 2t)]^2 \cdot \frac{1}{1-2t} \cdot \frac{d}{dt} (1 - 2t) = 3[\ln(1 - 2t)]^2 \cdot \frac{1}{1-2t} \cdot (-2) = -\frac{6}{1-2t} \ln^2(1 - 2t) \end{aligned}$$

Exercise Evaluate and simplify the following derivatives.

- $\frac{d}{dx} \{2[3 + \cot(4x)]^3\}$  [Answer:  $-24[3 + \cot(4x)]^2 \csc^2(4x)$ ]
- $\frac{d}{dx} \sqrt{1 + \cos(2x)}$  [Answer:  $\frac{\sin(2x)}{\sqrt{1+\cos(2x)}}$ ]
- $\frac{d}{dt} [15 \cos^2(120\pi t)]$  [Answer:  $-3600\pi \sin(120\pi t) \cos(120\pi t)$ ]
- $\frac{d}{dx} [(e^{1/x})^2]$  [Answer:  $-\frac{2}{x^2} e^{2/x}$ ]
- $\frac{d}{du} [\sec(2u) + \tan(2u)]^3$  [Answer:  $6 \sec(2u) [\sec(2u) + \tan(2u)]^3$ ]
- $\frac{d}{dx} \sqrt{x^2 + 1}$  [Answer:  $\frac{x}{\sqrt{x^2+1}}$ ]
- $\frac{d}{dx} \frac{1}{(1-4x)^5}$  [Answer:  $\frac{20}{(1-4x)^6}$ ]

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- The electric resistance  $R$  of a wire varies inversely as the square of its radius  $r$ . For a given wire,  $R = 4.66\Omega$  for  $r = 0.15\text{mm}$ . Find the derivative of  $R$  with respect to  $r$  for this wire.

[Answer:  $\frac{dR}{dr} = -\frac{0.21}{r^3}$ ]

### Derivative of Products and Quotients

$\frac{d}{dx}(uv) = u \frac{d}{dx}(v) + v \frac{d}{dx}(u)$	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{d}{dx}(u) - u \frac{d}{dx}(v)}{v^2}$
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**Example** Evaluate  $\frac{d}{dx} (3e^{4x} + 4x^2 \ln x)^2$ .

**Solution:**

(First apply the general power rule: power rule + chain rule)

$$\frac{d}{dx} (3e^{4x} + 4x^2 \ln x)^2 = 2(3e^{4x} + 4x^2 \ln x) \frac{d}{dx} (3e^{4x} + 4x^2 \ln x)$$

(Apply the product rule for the derivative of the second term)

$$\begin{aligned} &= 2(3e^{4x} + 4x^2 \ln x) \cdot \left[ 3e^{4x} \cdot \frac{d}{dx}(4x) + (4x^2) \frac{d}{dx} \ln x + (\ln x) \frac{d}{dx}(4x^2) \right] \\ &= 2(3e^{4x} + 4x^2 \ln x) \cdot \left[ 3e^{4x} \cdot 4 + (4x^2) \cdot \frac{1}{x} + (\ln x) \cdot (8x) \right] \\ &= 2(3e^{4x} + 4x^2 \ln x) \cdot [12e^{4x} + 4x + 8x \ln x] \end{aligned}$$

(Factor out common factor 4 from the three terms in the last factor)

$$= 8(3e^{4x} + 4x^2 \ln x) \cdot [3e^{4x} + x + 2x \ln x]$$

**Example** Evaluate  $\frac{d}{dt} \ln \left( \sqrt[3]{\frac{t}{t^3+1}} \right)$

**Solution:**

(Apply Chain Rule)  $\left( \frac{d}{du} \ln u \Big|_{u=\sqrt[3]{\frac{t}{t^3+1}}} \right) \cdot \left( \frac{d}{du} \sqrt[3]{u} \Big|_{u=\frac{t}{t^3+1}} \right) \cdot \left( \frac{d}{dt} \frac{t}{t^3+1} \right)$

(Apply basic differentiation formulae for the first two and the quotient rule for the last)

$$\begin{aligned} &= \left( \frac{1}{u} \Big|_{u=\sqrt[3]{\frac{t}{t^3+1}}} \right) \cdot \left( \frac{1}{3} u^{-2/3} \Big|_{u=\frac{t}{t^3+1}} \right) \cdot \frac{(t^3+1)(1)-(t)(3t^2)}{(t^3+1)^2} \\ &= \left( \frac{t}{t^3+1} \right)^{-1/3} \cdot \frac{1}{3} \left( \frac{t}{t^3+1} \right)^{-2/3} \cdot \frac{t^3+1-3t^3}{(t^3+1)^2} = \frac{1}{3} \left( \frac{t}{t^3+1} \right)^{-1} \frac{1-2t^3}{(t^3+1)^2} \\ &= \frac{(t^3+1)(1-2t^3)}{3t(t^3+1)^2} = \frac{1-2t^3}{3t(t^3+1)} \end{aligned}$$

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**Exercise** Evaluate and simplify the following derivatives.

- $\frac{d}{dx} [2x^3(3 - x^3)^4]$  [Answer:  $6x^2(3 - 5x^3)(3 - x^3)^3$ ]
- $\frac{d}{dx} [x^3 \cos(x^2 + 1)]$  [Answer:  $3x^2 \cos(x^2 + 1) - 2x^4 \sin(x^2 + 1)$ ]
- $\frac{d}{dt} [t \csc^3(2t)]$  [Answer:  $\csc^3(2t) [1 - 6t \cot(2t)]$ ]
- $\frac{d}{d\theta} [e^{-2\theta} \sin(3\theta)]$  [Answer:  $e^{-2\theta} [3 \cos(3\theta) - 2 \sin(3\theta)]$ ]
- $\frac{d}{dx} (e^{4x} \cos x)$  [Answer:  $e^{4x} (4 \cos x - \sin x)$ ]
- $\frac{d}{dt} \left( e^t \cos \frac{3t}{4} \right)$  [Answer:  $e^t \left( \cos \frac{3t}{4} - \frac{3}{4} \sin \frac{3t}{4} \right)$ ]
- $\frac{d}{dx} [e^{x^3} \cos(2x)]$  [Answer:  $e^{x^3} [3x^2 \cos(2x) - 2 \sin(2x)]$ ]
- $\frac{d}{dx} (x e^{\tan x})$  [Answer:  $e^{\tan x} (1 + x \sec^2 x)$ ]
- $\frac{d}{dt} [(t^2 + 1) \tan^{-1} t - t]$  [Answer:  $2t \tan^{-1} t$ ]
- $\frac{d}{du} \left( u \sin^{-1} 2u + \frac{1}{2} \sqrt{1 - 4u^2} \right)$  [Answer:  $\sin^{-1} 2u$ ]
- $\frac{d}{dx} [2x(5 - 3x)^4]$  [Answer:  $10(1 - 3x)(5 - 3x)^3$ ]
- $\frac{d}{dy} \frac{2y}{1 - \cot 3y}$  [Answer:  $\frac{2 - 2 \cot 3y - 6y \csc^2 3y}{(1 - \cot 3y)^2}$ ]
- Find the derivative of  $y = \frac{3x^2 + x}{1 - 4x}$  at  $(2, -2)$ . [Answer:  $-\frac{5}{7}$ ]
- Evaluate the derivative of  $y = \frac{x}{\sqrt{1 - 4x}}$  for  $x = -2$ . [Answer:  $\frac{5}{27}$ ]

### Differentiation of Implicit Functions

To find the derivative  $dy/dx$  when  $y$  is defined as an implicit function of  $x$ , we differentiate each term of the equation with respect to  $x$ , regarding  $y$  as a differentiable function of  $x$ .

**Example** Given  $y^2 + xy = \sin(x + y)$ , find  $\frac{dy}{dx}$ .

**Solution:**

(Since we cannot use the given equation to solve for  $y$ , that is, isolate  $y$  by itself and express  $y$  in terms of  $x$  alone, we use implicit differentiation and differentiate both sides

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of the equation with respect to  $x$ , keeping in mind that  $y$  is an implicit function of  $x$ )

$$\frac{d}{dx}[y^2 + xy] = \frac{d}{dx}\sin(x + y)$$

(For the left hand side, we differentiate the first term by using the chain rule and the second term by applying the product rule; for the right hand side, we use the chain rule)

$$2y \frac{dy}{dx} + \left(\frac{dx}{dx} \cdot y + x \cdot \frac{dy}{dx}\right) = \cos(x + y) \cdot \left(\frac{dx}{dx} + \frac{dy}{dx}\right)$$

$$\Rightarrow 2y \frac{dy}{dx} + \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) = \cos(x + y) \cdot \left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow 2y \frac{dy}{dx} + \left(y + x \cdot \frac{dy}{dx}\right) = \cos(x + y) \cdot \left(1 + \frac{dy}{dx}\right)$$

(Expand the equation and rearrange the terms so that all the terms involving  $\frac{dy}{dx}$  appear on the left hand side)

$$2y \frac{dy}{dx} + y + x \frac{dy}{dx} = \cos(x + y) + \cos(x + y) \frac{dy}{dx}$$

$$\Rightarrow 2y \frac{dy}{dx} + x \frac{dy}{dx} - \cos(x + y) \frac{dy}{dx} = \cos(x + y) - y$$

$$\text{or } y - \cos(x + y) = \cos(x + y) \frac{dy}{dx} - x \frac{dy}{dx} - 2y \frac{dy}{dx}$$

(Factor and solve for  $\frac{dy}{dx}$ )

$$\Rightarrow [2y + x - \cos(x + y)] \frac{dy}{dx} = \cos(x + y) - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos(x+y)-y}{2y+x-\cos(x+y)}$$

$$\text{or } y - \cos(x + y) = [\cos(x + y) - x - 2y] \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-\cos(x+y)}{\cos(x+y)-x-2y}$$

### Exercise

- Given  $x^2 + y^4 = \sin(x + y)$ , find  $\frac{dy}{dx}$ .

$$[\text{Answer: } \frac{dy}{dx} = \frac{\cos(x+y)-2x}{4y^3-\cos(x+y)} \text{ or } \frac{dy}{dx} = \frac{2x-\cos(x+y)}{\cos(x+y)-4y^3}]$$

- Given  $\cot(2x) - 3 \csc(xy) = y^2$ , find  $\frac{dy}{dx}$ . [Answer:  $\frac{dy}{dx} = \frac{2 \csc^2(2x) - 3y \csc(xy) \cot(xy)}{3x \csc(xy) \cot(xy) - 2y}$ ]

- Find  $\frac{dy}{dx}$  if  $3y^4 + xy^2 + 2x^3 - 6 = 0$ . [Answer: ]

- Find  $\frac{dy}{dx}$  if  $2x^3y + (x + y^2)^3 = x^4$ . [Answer:  $\frac{dy}{dx} = \frac{4x^3 - 6x^2y - 3(x + y^2)^2}{2x^3 + 6y(x + y^2)^2}$ ]

- Find  $\frac{dy}{dx}$  if  $(1 + y^2)^3 - x^2y = 7x$ . [Answer:  $\frac{dy}{dx} = \frac{7 + 2xy}{6y(1 + y^2)^2 - x^2}$ ]



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- Find the derivative  $\frac{dy}{dx}$  if  $y \sec(2x) = \sin^{-1}(3y)$ . [Answer:  $\frac{dy}{dx} = \frac{2y\sqrt{1-9y^2} \sec(2x) \tan(2x)}{3-\sqrt{1-9y^2} \sec(2x)}$ ]

### Higher Derivatives

**Example** Find the second derivative of  $y = \frac{2x}{3x+2}$ .

**Solution:**

(Apply the quotient rule to find the first derivative)

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x+2) \frac{d}{dx}(2x) - (2x) \frac{d}{dx}(3x+2)}{(3x+2)^2} = \frac{(3x+2) \frac{d}{dx}(2x) - (2x) \frac{d}{dx}(3x+2)}{(3x+2)^2} \\ &= \frac{(3x+2)(2) - (2x)(3)}{(3x+2)^2} = \frac{6x+4-6x}{(3x+2)^2} = \frac{4}{(3x+2)^2}\end{aligned}$$

(To find the second derivative, apply either the quotient rule one more time, or more simply the general power rule]

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{4}{(3x+2)^2} \right) = \frac{d}{dx} [4 \cdot (3x+2)^{-2}] \\ &= 4(-2)(3x+2)^{-3} \frac{d}{dx}(3x+2) = 4(-2)(3x+2)^{-3}(3) = -\frac{24}{(3x+2)^3}\end{aligned}$$

**Example** Find  $y''$  for the implicit function defined by  $2x^2 + 3y^2 = 6$ .

**Solution:**

(Use implicit differentiation: differentiate both sides with respect to  $x$ )

$$2(2x) + 3(2y) \frac{dy}{dx} = 0 \Rightarrow 4x + 6y \frac{dy}{dx} = 0 \Rightarrow 2x + 3y \frac{dy}{dx} = 0$$

(Use implicit differentiation the second time to find  $y''$ )

$$2(1) + 3 \left[ \left( \frac{dy}{dx} \right) \left( \frac{dy}{dx} \right) + y \frac{d^2y}{dx^2} \right] = 0 \Rightarrow 2 + 3 \left[ \left( \frac{dy}{dx} \right)^2 + y \frac{d^2y}{dx^2} \right] = 0$$

(Substitute the result for  $\frac{dy}{dx}$  from the first equation into the second equation and solve for

$$\text{the second derivative) } 2x + 3y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{3y}$$

$$\Rightarrow 2 + 3 \left[ \left( -\frac{2x}{3y} \right)^2 + y \frac{d^2y}{dx^2} \right] = 0 \Rightarrow 2 + \frac{4x^2}{3y^2} + 3y \frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{3y} \left( 2 + \frac{4x^2}{3y^2} \right)$$

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(Simplify the result by using the original given equation)

$$2x^2 + 3y^2 = 6 \Rightarrow 2x^2 = 6 - 3y^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{3y} \left[ 2 + \frac{2(2x^2)}{3y^2} \right] = -\frac{1}{3y} \left[ 2 + \frac{2(6-3y^2)}{3y^2} \right] = -\frac{1}{3y} \cdot \frac{6y^2 + 12 - 6y^2}{3y^2} = -\frac{12}{9y^3} = -\frac{4}{3y^3}$$

### Exercise

- For  $f(x) = \sec x$ , find  $f' \left( \frac{\pi}{6} \right)$  and  $f'' \left( \frac{\pi}{6} \right)$ . [Answer:  $f' \left( \frac{\pi}{6} \right) = \frac{2}{3}$ ,  $f'' \left( \frac{\pi}{6} \right) = \frac{10}{9}\sqrt{3}$ ]
- Find the higher derivatives of  $f(x) = x(x^2 - 1)^2$ . [Answer:  $f'(x) = 5x^4 - 6x^2 + 1$ ,  
 $f''(x) = 20x^3 - 12x$ ,  $f'''(x) = 60x^2 - 12$ ,  $f^{(4)}(x) = 120x$ ,  $f^{(5)}(x) = 120$ ,  
 $f^{(n)}(x) = 0$  for  $n \geq 6$ ]
- Evaluate the second derivative of  $y = \frac{2}{1-x}$  for  $x = -2$ . [Answer:  $\frac{4}{27}$ ]