Mid - point Formula : $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right)$

Distance Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Circle centered at (h, k) with radius r: $(x - h)^2 + (y - k)^2 = r^2$

Example Find the center and the radius of the circle with equation $x^2 + y^2 - 9x - 6y + 8 = 0$.

Solution:

(Rewrite the equation in the standard form by first separating the terms in the equation according to the variables)

$[x^2 - 9x] + [y^2 - 6y] + 8 = 0$

(Complete the Square)

$$\left[\left(x^2 - 9x + \left(\frac{-9}{2}\right)^2\right) - \left(\frac{-9}{2}\right)^2\right] + \left[\left(y^2 - 6y + \left(\frac{-6}{2}\right)^2\right) - \left(\frac{-6}{2}\right)^2\right] + 8 = 0$$

$$\Rightarrow \left[\left(x - \frac{9}{2}\right)^2 - \frac{81}{4}\right] + [(y - 3)^2 - 9] + 8 = 0$$

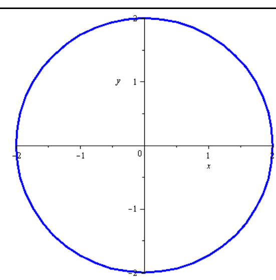
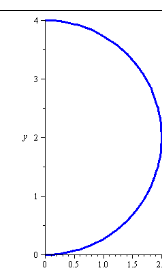
(Collect like terms and write the equation in standard form)

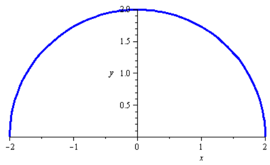
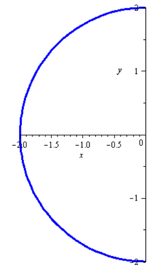
$$\Rightarrow \left(x - \frac{9}{2}\right)^2 - \frac{81}{4} + (y - 3)^2 - 9 + 8 = 0 \Rightarrow \left(x - \frac{9}{2}\right)^2 + (y - 3)^2 = \frac{85}{4}$$

Center = $\left(\frac{9}{2}, 3\right)$, **radius =** $\sqrt{\frac{85}{4}} = \frac{\sqrt{85}}{2}$

Exercise

- Find the equation of a circle which has a diameter connecting the two points $(2, -3)$ and $(6,4)$. [Answer: $x^2 + y^2 - 8x - y = 0$]
- Sketch the graph of the following equation.

Equation	Answer	Equation	Answer
$x^2 + y^2 = 4$		$x = \sqrt{4y - y^2}$	

Equation	Answer	Equation	Answer
$y = \sqrt{4 - x^2}$		$x = -\sqrt{4 - y^2}$	
$x^2 + y^2 = 2y$			

<u>Parabola</u>:

- vertex at (h, k) and axis parallel to the *x*-axis: $(y - k)^2 = 4p(x - h)$
- vertex at (h, k) and axis parallel to the *y*-axis: $(x - h)^2 = 4p(y - k)$

<u>Example</u> Find the vertex and the *x*- and *y*-intercepts of the parabola $x^2 + y = 6x$.

Sketch the graph.

Solution:

(Set $y = 0$ and solve for *x* for the *x*-intercept(s)) $y = 0$: $x^2 = 6x \Rightarrow x^2 - 6x = 0$

$\Rightarrow x(x - 6) = 0 \Rightarrow x = 0$ or $x - 6 = 0 \Rightarrow \boldsymbol{x = 0}$ **or** $\boldsymbol{x = 6}$ (*x*-intercepts)

(set $x = 0$ and solve for *y* for the *y*-intercept)

$x = 0$: $0^2 + y = 6(0) \Rightarrow \boldsymbol{y = 0}$ (*y*-intercept)

(To find the vertex, we need to use the method of completing the square to rewrite the equation in the standard form; first we identify the term involving the square of a variable in the equation, rearrange the terms in the equation so that all terms involving that variable are on the same side of the equation) $x^2 - 6x + y = 0$

(Complete the square)

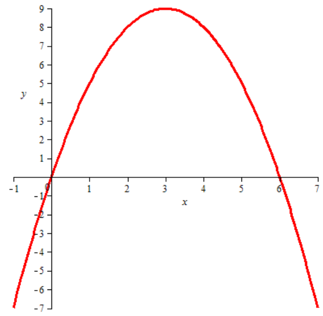
$$\left[x^2 - 6x + \left(\frac{-6}{2}\right)^2\right] - \left(\frac{-6}{2}\right)^2 + y = 0 \Rightarrow [x^2 - 6x + 9] - 9 + y = 0$$

$$\Rightarrow (x - 3)^2 - 9 + y = 0$$

(Write the equation in standard form and identify the vertex)

$\Rightarrow y - 9 = -(x - 3)^2$ **Vertex** $= \boldsymbol{(3, 9)}$

(Sketch the graph)



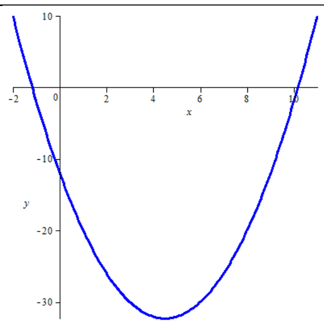
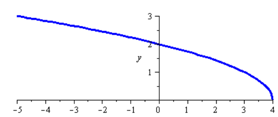
Exercise

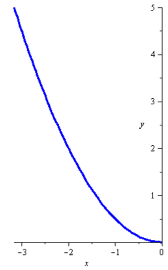
- Find the vertex of the parabola and sketch the graph.
  - $y^2 - 2y - 8x - 31 = 0$ [Answer: vertex: $(-4,1)$, focus: $(-2,1)$, directrix: $x = -6$]
  - $x^2 + 6x + 4y + 5 = 0$ [Answer: vertex: $(-3,1)$, focus: $(-3,0)$, directrix: $y = 2$]
- Find the vertex, the axis of symmetry, and the maximum or minimum value of the function, then graph the function.
  - $y = x^2 + 10x + 23$

    [Answer: Vertex: $(-5,-2)$, axis of symmetry: $x = -5$, minimum value: –2]
  - $y = \frac{1}{2}x^2 - 4x + 8$

    [Answer: Vertex: $(4,0)$, axis of symmetry: $x = 4$, minimum value: 0]
  - $y = -2x^2 + 10x - \frac{23}{2}$

    [Answer: Vertex: $\left(\frac{5}{2},1\right)$, axis of symmetry: $x = \frac{5}{2}$, maximum value: 1]
- Sketch the graph of the following equation.

Equation	Answer	Equation	Answer
$y = x^2 - 9x - 12$		$y = \sqrt{4 - x}$	

Equation	Answer	Equation	Answer
$x = -\sqrt{2y}$			

Ellipse $(a > b)$:

- centered at (h, k) with major axis parallel to the x-axis: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
- centered at (h, k) with major axis parallel to the y-axis: $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$

Example Sketch the solution set of the inequality $4 - x^2 - 4y^2 \geq 0$

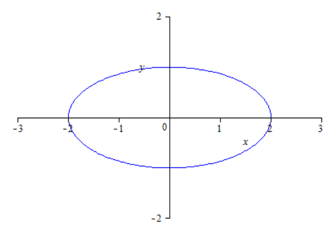
Solution:

(Replace the inequality by an equation, which defines a curve representing the boundary of the solution set we are looking for)

$$4 - x^2 - 4y^2 = 0 \Rightarrow x^2 + 4y^2 = 4 \Rightarrow$$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

(This is an ellipse centered at the origin with horizontal major axis of length $2a = 4$ and vertical minor axis of length $2b = 2$.)



(Pick a test point from each region and check whether the coordinates of the test point satisfy the inequality; if it does, all points in the same region as that test point is part of the solution set; otherwise, the region that the test point belongs to is not part of the solution set)

Test point (0,0): $4 - 0^2 - 4(0^2) \geq 0?$ Yes!

Test point (3,0): $4 - 3^2 - 4(0^2) \geq 0?$ No!

Moreover, since the points on the ellipse satisfy the equation $4 - x^2 - 4y^2 = 0$, they also satisfy the inequality $4 - x^2 - 4y^2 \geq 0$.

Hence **the solution set of the inequality contains all the points inside the ellipse, including the points on the ellipse**.

Hyperbola $(a > b)$:

- centered at (h, k) with foci (and vertices) on a horizontal line (same *y*-coordinates):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

- centered at (h, k) with foci (and vertices) on a vertical line (same *x*-coordinates):

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Exercise Find the center and the vertices of the hyperbola $4y^2 - x^2 + 24y + 4x + 28 = 0$, and then sketch the graph.

[Answer: center: $(2, -3)$, vertices: $(2, -4), (2, -2)$]