Binomial Theorem

Binomial Theorem: For any non-negative integer n,

$(a+b)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}b^r$

$= C(n,0)a^n + C(n,1)a^{n-1}b + \cdots + C(n,r)a^{n-r}b^r + \cdots + C(n,n-1)ab^{n-1} + C(n,n)b^n$

where $C(n,r) = \frac{n!}{r!(n-r)!}$.

Note: One can also get the values for $C(n,r)$ by using the Pascal's triangle:

- The first row has a single entry of 1; the number of entries increases by one as we move from any row to the next row
- Starting with the second row, the first and last entries of the row are always 1; any other entry in a row is equal to the sum of the nearest two entries in the previous row.

$$\begin{array}{lccccccccc}
(n=0) & & & & & 1 & & & & \\
(n=1) & & & & 1 & & 1 & & & \\
(n=2) & & & 1 & & 2 & & 1 & & \\
(n=3) & & 1 & & 3 & & 3 & & 1 & \\
(n=4) & 1 & & 4 & & 6 & & 4 & & 1 \\
\vdots & & & & & \vdots & & & &
\end{array}$$

Example Expand $(3k+1)^2$.

Solution:

(Since the power $n=2$ is relatively small, we can simply multiply out directly using FOIL; an alternative method is to apply the Binomial Theorem with $a=3k, b=1, n=2$)

(Method 1: Use the Square of a Sum Formula $(a+b)^2 = a^2+2ab+b^2$)

$(3k+1)^2 = (3k)^3 + 2(3k)(1) + (1)^2 = \mathbf{9k^2+6k+1}$

(Method 2: Multiply out directly using FOIL)

$(3k+1)^2 = (3k+1)(3k+1) = 9k^2+3k+3k+1 = \mathbf{9k^2+6k+1}$

(Method 3: Apply the Binomial Theorem)

$(3k+1)^2 = \sum_{r=0}^{2} C(2,r)(3k)^{2-r}(1)^r = \sum_{r=0}^{2} C(2,r)(3k)^{2-r}$

$= C(2,0)(3k)^2 + C(2,1)(3k)^1 + C(2,2)(3k)^0$

$= \frac{2!}{0!2!}(9k^2) + \frac{2!}{1!1!}(3k) + \frac{2!}{2!0!}(1) = \frac{2}{(1)(2)}(9k^2) + \frac{2}{(1)(1)}(3k) + \frac{2}{(2)(1)}$

$= 9k^2 + 2(3k) + 1 = \mathbf{9k^2+6k+1}$

Exercise Without using a calculator,

- expand $(x^2-2y)^5$. [Answer: $x^{10} - 10x^8y + 40x^6y^2 - 80x^4y^3 + 80x^2y^4 - 32y^5$]
- expand $\left(\frac{2}{x} + 3\sqrt{x}\right)^4$. [Answer: $\frac{16}{x^4} + \frac{96}{x^{5/2}} + \frac{216}{x} + 216x^{1/2} + 81x^2$]

Example Find the coefficient of x^5y^7 in the expansion of $(2x-3y)^{12}$.

Solution:

(Since the power $n = 12$ is relatively large, multiplying out directly is tedious and prompt for errors; instead, we can apply the Binomial Theorem, which states that the term in the expansion involving x^5y^7 is the 8^{th} term $[12 - r = 5 \Longrightarrow r = 7]$

$C(12,7) \times (2x)^5 \times (-3y)^7 = C(12,7)(2)^5(-3)^7x^5y^7)$

$C(12,7)(2)^5(-3)^7 = \frac{12!}{7!5!}(2^5)[-(3^7)] = -\frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1}(2^5)(3^7) = \mathbf{-55{,}427{,}328}$

Exercise (You can use a regular scientific non-graphing non-programmable calculator)

- Find the 5^{th} term in the expansion of $(2x-5y)^6$. [Answer: $37{,}500x^2$]
- Find the 8^{th} term in the expansion of $(3x-2)^{10}$. [Answer: $-414{,}720x^3$]